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# Heisenberg groups—the fundamental ingredient to describe information, its transmission and quantization

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**Abstract**

For a singularity free gradient field in an open set of an oriented Euclidean space of dimension three we define a natural principal bundle out of an immanent complex line bundle. The fibres of this bundle encode information. The elements of both bundles are called internal variables. Several other natural bundles are associated with the principal bundle and, in turn, determine the vector field. Two examples are given and it is shown that for a constant vector field circular polarized waves with values in the principal bundle are associated with the vector field. These waves transmit information encoded in internal variables and, moreover, determine a Schrödinger representation. On $SU(2)$ a relation between spin representations and Schrödinger representations is established. The link between the spin $\frac{1}{2}$ model and the Schrödinger representations yields a connection between a microscopic and a macroscopic viewpoint. Quantization and its link to information is derived out of the Schrödinger representation.

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## 1. Introduction

Our primary interest in these notes is the concepts of information and information transmission. We will present an approach which allows us to link both notions in a natural way. The technical devices will be the concepts of Heisenberg groups and their Lie algebras, i.e. Heisenberg algebras. Two notions of information will be applied and combined: on the one hand the pointal notion of information and on the other hand the probabilistic concept of information.

At first, let us consider a simple but basic situation, namely a vector field in Euclidean three-space. Given an integral curve $\beta$ of it, i.e. a field line, we may consider a point on $\beta$ as a piece of information. The parametrization of $\beta$ determines a motion of an initial point with a velocity determined by $\dot{\beta}$ and hence by the vector field. So the motion caused by the vector field describes the transmission of information in a universal time. A more elaborate setting for the treatment of information and its transmission in connection with a vector field is based on the following observations.

Each singularity free gradient field in an open part of a three-dimensional Euclidean space admits a variety of natural bundles such as complex line bundles, principal bundles, bundles of complex numbers, quaternions, Heisenberg groups, Heisenberg algebras etc. All these bundles emanate from the fact that a field vector (i.e. the value of the vector field at a point) has a real plane as an orthogonal complement in three-space causing a Heisenberg algebra bundle. These planes form a complex line bundle in a natural way. All other bundles mentioned can be constructed out of this complex line bundle. Each one of them determines the vector field uniquely. This is why the elements of the respective bundles are called internal variables of the vector field.

These bundles are trivial when restricted to an integral curve $\beta$; hence one fibre is enough to describe the bundle along $\beta$. Each of the fibres can be regarded as an initial fibre and therefore its elements as initial internal variables. Let us confine ourselves to the complex line bundle for the moment. It contains a natural principal bundle $\mathcal{P}^a$ the fibres of which are circles.

We will demonstrate on two examples, namely on the one hand on the constant vector field and on the other hand on the $m/r$-gradient field (here called the solar field), a possible interpretation of internal variables and a use of them. We will show that circular polarized electromagnetic waves travelling along a field line can be regarded as consisting of internal variables.

The transmission of information along vector fields follows a motion on geodesics on the principal bundle $\mathcal{P}^a$ restricted to a field line $\beta$, i.e. on $\mathcal{P}^a|_\beta$. In that sense a piece of information is transmitted along a field line. This line serves as a channel of information. In the case of a constant vector field or in the case of the solar field the rotation of a geodesic motion about the channel of information yields a geometric picture of a wave. Hence it can naturally be associated with the motion of a point along the integral curve. This kind of a wave can, for example, be interpreted as an electromagnetic wave.

Such a wave, in fact, allows us to construct a Heisenberg group, which admits a Schrödinger representation of any frequency. It implements the geometry of the Heisenberg group into the unitary group of a Hilbert space of signals. The infinitesimal Schrödinger representation carries over the geometry of the Heisenberg algebra into the Lie algebra of the unitary group of that Hilbert space, yielding in particular the uncertainty relation of Heisenberg. Thus the information encoded in a complex line, i.e. a Heisenberg group modulo its centre, as well as the information transmission along the centre (the information channel) are equivalently described by the Schrödinger representation. It modulates the information contained in the Heisenberg algebra on signals.

Quite another bundle of internal variables is the bundle of quaternions, a trivial bundle which contains the trivial $SU(2)$-bundle. The (not necessarily trivial) bundle of Heisenberg groups is closely related to this $SU(2)$-bundle. This is due to the following considerations.

On $SU(2)$ there is a direct relation between spin representations and Schrödinger representations, in particular for spin $\frac{1}{2}$. The remarkable fact that the group structure of $SU(2)$ is determined by only one Heisenberg group is an observation which is crucial for instance in the imaging principles of magnetic resonance imaging (MRI). The reason is that the information encoded in the spins can be analysed via signals by the ambiguity function, which is determined by the Schrödinger representation. This is an implementation of the fact that the Schrödinger representation causes a spin $\frac{1}{2}$ representation and vice versa. Hence we have a correspondence which gives rise to a spin model that, on the microscopic level, encodes information which can be read out on the macroscopic level by the ambiguity function. Thus, through the relation between the spin $\frac{1}{2}$ formalism and the Schrödinger representation, we have the mathematical basis for a link from the microscopic to the macroscopic scale. In MRI a technical realization of the spin model described here uses the quantum mechanical spin of protons which interacts with constant magnetic fields and signals.

This spin $\frac{1}{2}$–Schrödinger correspondence is ultimately connected to the three dimensions of the Euclidean space surrounding us.

We finally link information and its transmission formulated via the Schrödinger representation to a natural and established notion of quantization. We motivate it by entropy preserving transformations. Therefore, this kind of quantization is an aspect of information theory expressed by the theory of geometric optics.

Most of the results in this paper require rather involved proofs and hence many proofs will only be outlined. More detailed versions can be found in Binz *et al* (2003) and in Pods (2003).

We are in debt to the referees for a series of valuable remarks.

## 2. Gradient fields in three-space and internal variables

Any singularity free smooth gradient field $X : O \to O \times E = TO$ defined on an open set $O$ in an oriented three-dimensional Euclidean space $E$ admits a natural collection of internal variables. Here $TO = O \times E$ is the tangent bundle of $O$. The orthogonal complement $F^{a(x)} := (a(x))^{\perp} \subset E$ is a two-dimensional plane for any $x \in O$, where the smooth map $a : O \to E$ is the principal part of $X = (\mathrm{id}, a)$. It is the basic collection of internal variables of $X$ at $x$. One of the fundamental observations in these notes is that the $\mathbb{R}$-linear space $F^{a(x)}$ is a $\mathbb{C}$-linear space in a natural way, as well. In order to make this apparent, we need to embed $F^{a(x)}$ into a larger space, namely into the skew field of quaternions $\mathbb{H} = \mathbb{R} \cdot e \oplus E$ generated by $E$ with $e$ the unit element. Let $a(x) = b$ for reasons of simplicity. The key to the $\mathbb{C}$-linear structure of $F^b$ lies in the 2–2-splitting of $\mathbb{H}$ as

$$\mathbb{H} = \mathbb{C}^b \oplus F^b$$

where $F^b$ is the orthogonal complement of $b$ in $E$ and $\mathbb{C}^b$ is the orthogonal complement of $F^b$ in $\mathbb{H}$. The scalar product $\langle\, ,\rangle$ on $E$ extends to all of $\mathbb{H}$ such that $e \in \mathbb{H}$ is a unit vector orthogonal to $E$. The orthogonal complement $\mathbb{C}^b$ of $F^b$ is a commutative subfield of $\mathbb{H}$ generated by $e$ and $\frac{b}{|b|}$. It is naturally isomorphic to the field $\mathbb{C}$. The isomorphism maps 1 on $e$ and i on the *imaginary unit* $\frac{b}{|b|}$, respectively. The unit circle in $\mathbb{C}^b$ forms the unitary group of $\mathbb{C}^b$, which, of course, is isomorphic to $U(1)$. We will call it $U^{\frac{b}{|b|}}(1)$ in the following. Hence the vector $\frac{a(x)}{|a(x)|}$ has turned into an imaginary unit for any $x \in O$.

The product $u \cdot v$ in $\mathbb{H}$ for $u, v \in E$ is given by

$$u \cdot v = u \times v - \langle u, v\rangle \cdot e \tag{1}$$

where $\times$ is the cross product in $E$. The centre of $\mathbb{H}$ is $\mathbb{R} \cdot e$. As can be easily seen, $\mathbb{C}^b$ operates on $F^b$ by multiplication from the right. Hence $F^b$ is a $\mathbb{C}^b$-linear space and, in turn, a $\mathbb{C}$-vector space, as well.

The real plane $F^b$ carries a symplectic structure $\omega^b$, too, defined by

$$\omega^b(h_1, h_2) := \langle h_1 \times b, h_2\rangle \qquad \forall h_1, h_2 \in F^b. \tag{2}$$

This symplectic structure is based on the rotation by $\frac{\pi}{2}$ given by the multiplication with the imaginary unit $\frac{b}{|b|}$.

$O \times \mathbb{H}$ is a trivial bundle containing $\bigcup_{x\in O}\{x\} \times F^{a(x)}$. This set, regarded as a topological subspace of $O \times \mathbb{H}$, inherits the structure of a smooth vector bundle over $O$. Since each fibre is a complex line, it is in fact a complex line bundle called $\mathbb{F}^a$. Accordingly, $\cup_{x\in O}\mathbb{C}^{a(x)} \subset \mathbb{H}$ is the total space of a bundle over $O$, the bundle of complex numbers associated with $X$.

The bundles $\mathbb{F}^a$ and $\mathbb{C}^a$ are collections of internal variables of $X$. While $\mathbb{C}^a$ is a trivial bundle, $\mathbb{F}^a$ is not, in general. Both sorts of internal variables of $X$ serve for the description of quite different qualities of $X$. The trivial bundle $O \times \mathbb{H}$ subsumes both $\mathbb{F}^a$ and $\mathbb{C}^a$.

The three-sphere $S^3 \subset \mathbb{H}$ can be regarded as a subgroup of $\mathbb{H}$ called $SU(2)$. Hence $O \times \mathbb{H}$ contains the trivial group bundle $O \times SU(2)$, which consists of a very important sort of internal variable of $X$ as we will see later on.

The complex line bundle $\mathbb{F}^a$ contains a natural principal bundle $\mathcal{P}^a$ characterizing $X$. The fibre at $x \in O$ is a circle of radius $|a(x)|^{-\frac{1}{2}}$. Another principal bundle $\dot{\mathbb{F}}^a$ of internal variables is obtained from $\mathbb{F}^a$ by deleting the zero section. The vector bundle $\mathbb{F}^a$ is an associated bundle for both $\mathcal{P}^a$ and $\dot{\mathbb{F}}^a$.

Moreover, there is a natural Heisenberg group bundle associated with $\mathcal{P}^a$, which determines $X$, too. To construct it, first of all we observe that for each $b \in S^2$ the unit circle $U^b(1) \subset \mathbb{C}^b$ is a commutative subgroup of $\mathbb{H}$. For any non-vanishing quaternion $h \in \mathbb{H}$ the circle $h \cdot U^b(1)$ inherits a group structure with $h$ as unit element; obviously, it is isomorphic to $U^b(1)$. For any $x \in O$ the singularity free vector field $X$ determines the set

$$G^{a(x)} := \frac{1}{\sqrt{|a(x)|}} \cdot U^{a(x)}(1) \oplus F^{a(x)} \subset \mathbb{H}.$$

It is a group under the multiplication

$$(z_1 + h_1) \cdot (z_2 + h_2) := z_1 \cdot z_2 \cdot \mathrm{e}^{\frac{1}{2}\omega^a(h_1,h_2)\frac{a}{|a|}} + h_1 + h_2 \tag{3}$$

holding for all $z_1, z_2 \in |a(x)|^{-\frac{1}{2}} \cdot U^{a(x)}(1)$ and any pair $h_1, h_2 \in F^{a(x)}$. Here the multiplication $z_1 \cdot z_2$ for $z_s = |a(x)|^{-\frac{1}{2}} \cdot \mathrm{e}^{\alpha_s \cdot \frac{a}{|a|}} \in |a(x)|^{-\frac{1}{2}} \cdot U^{a(x)}(1)$, $s = 1, 2$, is given by the multiplication in $U^{a(x)}(1)$ followed by a rescaling of the product by a factor $|a(x)|^{-\frac{1}{2}}$. This group $G^{a(x)}$ is a Heisenberg group. The topological subspace

$$\mathbb{G}^a := \bigcup_{x\in O}\{x\} \times G^{a(x)}$$

of $O \times \mathbb{H}$ is a smooth group bundle, the Heisenberg group bundle of $X$ (cf Pods 2003). Obviously, $\mathbb{G}^a$ also manifests a collection of internal variables which is of particular importance in particle physics and imaging techniques. The latter will be indicated below. Clearly, $\mathbb{G}^a$ determines $X$ uniquely; the value of $X$ at $x$ is the tangent vector at the unity of the centre $|a(x)|^{-\frac{1}{2}} \cdot U^{a(x)}(1)$ of $G^{a(x)}$. Moreover, $\mathbb{F}^a$ together with the fibrewise given symplectic forms determines $X$ and so does the fibrewise oriented principal bundle $\mathcal{P}^a$.

## 3. Two examples

If we consider specific vector fields in these notes, most of the time we will concentrate on the two types presented in more detail in this section. In doing so, we can demonstrate the principles of our construction of internal variables in a rather simple setting while at the same time we are working with vector fields which are important in MRI and classical mechanics, respectively. Thus the notion of internal variables allows interpretations in well-known surroundings.

First let us consider a constant vector field $X$ on $O \subset \dot{E}$ with principal part having value $a \in E$ for all $x \in O$. Obviously the principal bundle $\mathcal{P}^a$ is trivial, i.e. $\mathcal{P}^a$ is isomorphic to $O \times U(1)$.

Since an integral curve $\beta$ of $X$ is a straight line segment parametrized by

$$\beta_{x_0}(t) = t \cdot a + x_0 \qquad \text{with} \quad \beta_{x_0}(0) = x_0$$

the restriction $\mathcal{P}^a\big|_{\beta_{x_0}}$ of $\mathcal{P}^a$ to the image of $\beta_{x_0}$ is a cylinder with radius $|a|^{-\frac{1}{2}}$. The image of an integral curve as a point set is a straight line segment, as well.

As the second type of an example of a principal bundle $\mathcal{P}^a$ associated with a singularity free vector field, let us consider a central symmetric gradient field $X = \operatorname{grad} V$ of a smooth function $V$ on $O := E\backslash\{0\}$. This is to say that the potential $V$ and the principal part $a$ are invariant under $SO(E)$; the field lines are straight lines emerging from $0 \in E$. The level surfaces of the potential are spheres $S_r^2$ of radius $r$ centred at $\{0\}$. Obviously $a(x) = \pm|a(x)| \cdot \frac{x}{|x|}$ for all $x \in O$. The Gaussian curvature $\kappa_{S_r^2}(x)$ of $S_r^2$ satisfies $\kappa_{S_r^2}(x) = \frac{1}{|x|^2}$ at any $x \in S_r^2$ and for all $r > 0$.

To determine the principal bundle $\mathcal{P}^a$ we first look at $\mathcal{P}^a\big|_{S_r^2}$, i.e. its restriction to $S_r^2$. Clearly

$$\mathcal{P}^a\big|_{S_r^2} \subset TS_r^2 \subset O \times E \qquad \text{or, fibrewise formulated,} \qquad \mathcal{P}_x^a \subset T_x S_r^2 = F^{a(x)}$$

at any $x \in O$. For each $u \in SU(2)$ the inner automorphism

$$\tau_u : \mathbb{H} \longrightarrow \mathbb{H} \qquad \tau_u(k) := u \cdot k \cdot u^{-1} \tag{4}$$

fixes $e$ and is an element of $SO(E)$; more precisely, $\tau : SU(2) \longrightarrow SO(E)$ is two-to-one. Given any two points $x, x_0 \in S_r^2$, there is a quaternion $u \in SU(2)$ such that

$$x = \tau_u(x_0) \qquad \text{and} \qquad \frac{a(x)}{|a(x)|} = \tau_u\left(\frac{a(x_0)}{|a(x_0)|}\right).$$

Therefore, $\mathcal{P}_x^a = \tau_u\big(\mathcal{P}_{x_0}^a\big) = \tau_u\big(v_{x_0}\big) \cdot \tau_u\big(|a(x_0)|^{-\frac{1}{2}} \cdot U_{x_0}^a(1)\big)$ for any fixed $v_{x_0} \in \mathcal{P}_{x_0}^a$. For the two-to-one smooth map

$$f : SU(2) \longrightarrow \mathcal{P}^a\big|_{S_r^2}$$

defined by $f(u) = \tau_u\big(v_{x_0}\big)$ for all $u \in SU(2)$ the projection $\mathrm{pr}_H := \mathrm{pr}^a \circ f$ maps $SU(2)$ smoothly onto $S_r^2 \subset O$. Here $\mathrm{pr}^a : \mathcal{P}^a \to O$ denotes the projection from $\mathcal{P}^a$ onto $O$. The fibration on $SU(2)$ caused by $\mathrm{pr}_H$ is called the Hopf fibration.

Thus the total space of the principal bundle $\mathcal{P}^a = \dot{\mathbb{R}} \cdot x_0 \times \mathcal{P}^a\big|_{S_r^2}$ over $O$ is the $\mathbb{Z}_2$-quotient of $\dot{\mathbb{H}} := \mathbb{H}\backslash\{0\}$. The quotient map $f : \dot{\mathbb{H}} \longrightarrow \mathcal{P}^a$ assigns the value $f(r \cdot u) = \big(r \cdot x_0, \tau_u\big(v_{x_0}\big)\big)$ to $r \cdot u \in S_r^2 \subset \mathbb{H}$ for all $r \in \dot{\mathbb{R}}$ and for all $u \in SU(2)$. The Hopf projection $\mathrm{pr}_H$ extends to all of $\dot{\mathbb{H}}$ by setting

$$\mathrm{pr}_H(r \cdot u) = r \cdot \tau_u(x_0) \qquad \forall r \in \dot{\mathbb{R}} \quad \text{and} \quad \forall u \in SU(2).$$

Hence $\dot{\mathbb{H}}$ is a $U(1)$-principal bundle with $\mathrm{pr}_H : \dot{\mathbb{H}} \longrightarrow O$ as its projection, here called the extended Hopf fibration. Therefore, we state (cf Binz and Schempp 2000):

**Proposition 1.** *The extended Hopf fibration of* $\dot{\mathbb{H}}$ *over* $O := E\backslash\{0\}$ *defined by the projection* $\mathrm{pr}_H$ *is the two-fold covering of the principal bundle* $\mathcal{P}^a$ *of the central symmetric field X.*

The above proposition visualizes the geometry of the level surfaces and the field lines in terms of the principal bundle $\mathcal{P}^a$. The Hopf fibration plays an important role in teleportation, planetary motion and the treatment of the magnetic monopole (cf Binz and Schempp 1999, 2000, 2003, Greub and Petry 1975).

Finally we will illustrate from a longitudinal point of view the principal bundle associated with the gradient field with potential $V_s$ given by

$$V_s(x) := -\frac{\bar{m}}{|x|} \qquad \forall x \in O$$

where $\bar{m}$ is a positive real. This potential governs planetary motions.

Since grad $V_s$ is the principal part $a$ of the gradient field

$$\operatorname{grad} V_s(x) = -\frac{\bar{m}}{|x|^2} \cdot \frac{x}{|x|} \qquad x \in O \tag{5}$$

(called the *solar field* in what follows), the radius of $\mathcal{P}^a_x$ is $|x| \cdot \bar{m}^{-\frac{1}{2}}$ at any $x \in O$.

Hence if $\beta_{x_0}$ denotes an integral curve passing through $x_0 \in O$, the (trivial) principal bundle $\mathcal{P}^a\big|_{\beta_{x_0}}$ is a cone (cf Pods 2003).

The minus sign in (5) can be absorbed by switching from the right action of $\mathbb{C}^a_x$ on $F^a$ to the left action, as will be done in what follows.

## 4. Horizontal and periodic lifts, information and its transmission along the vector field

An internal variable can be interpreted as a bit of information. In other words, information is inscribed by means of internal variables. Thus the fibres $F^{a(x)}$ and $\mathcal{P}^a_x$ can be regarded as collections of pieces of information at $x$.

In order to treat a transmission of information along vector fields, we will first of all see that $\mathcal{P}^a$ admits a natural connection (cf Sniatycki 1980, Binz *et al* 1988, Binz and Schempp 2001a). A tangent vector $\xi \in T_{v_x}\mathcal{P}^a$ is of the form $\xi = (x, v_x, h, \zeta)$ where $h$ is tangent to $x$ and $\zeta$ is tangent to $v_x$. Then the connection form $\alpha^a$ is defined by

$$\alpha^a(v_x, \xi) := \langle v_x \times a(x), \zeta\rangle \qquad \forall x \in O \quad \forall v_x \in \mathcal{P}^a_x \quad \text{and} \quad \forall \xi \in T_{v_x}\mathcal{P}^a$$

where $v_x$ varies in the fibre $\mathcal{P}^a_x$ of $\mathcal{P}^a$ at $x$. Denoting

$$\mathrm{Hor}_{v_x} := \ker \alpha(v_x, \ldots) \qquad \forall v_x \in \mathcal{P}^a_x$$

the set

$$\mathrm{Hor} := \bigcup_{\substack{v_x \in \mathcal{P}^a_x \\ x \in O}} \mathrm{Hor}_{v_x}$$

inherits from $T\mathcal{P}^a$ a natural manifold structure. In fact, it is a vector subbundle of $T\mathcal{P}^a$ for which

$$T\mathrm{pr}^a : \mathrm{Hor}_{v_x} \longrightarrow T_x O$$

is an isomorphism for all $x \in O$. Associated with it is a natural covariant derivative (cf Greub *et al* 1973, Binz *et al* 1988). The curvature $\Omega$, i.e. the exterior covariant derivative of $\alpha^a$, can be written in the form

$$\Omega(x; v, w) = \frac{\kappa(x)}{|a(x)|} \cdot \omega^{a(x)}(v, w) \qquad \forall v, w \in F^{a(x)}.$$

Here $\kappa(x)$ is the Gaussian curvature of a level surface $S$ passing through $x \in O$ and $|a(x)|$ is the field strength at $x$. Moreover, $\omega^{a(x)}$ at $x \in S$ is the symplectic structure of $F^{a(x)} = T_x S$. Clearly, $\frac{a(x)}{|a(x)|}$ is the unit normal of $S$ at $x$.

This shows that the curvature $\Omega$ at $x$ depends entirely on the geometry of the level surface $S$ of the potential of $X$, the field strength and the symplectic form.

Any (smooth) curve $\sigma$ in $O$ defined on an interval in $\mathbb{R}$ yields a unique *horizontal lift* $\sigma^{\text{hor}}$ through an initial point $v_x \in \mathcal{P}^a_x$, say. Horizontal means that

$$\dot{\sigma}^{\text{hor}}(t) \in \text{Hor}_{\sigma(t)} \qquad \forall t.$$

The horizontal lift $\sigma^{\text{hor}}$ of a (smooth) curve $\sigma$ in $O$ satisfies

$$\sigma^{\text{hor}}(t) \in \mathcal{P}^a_{\sigma(t)} \qquad \forall t.$$

This is to say that $\sigma^{\text{hor}}$ is a curve in the restriction $\mathcal{P}^a|_\sigma$ of $\mathcal{P}^a$ to $\sigma$. Obviously $\mathcal{P}^a|_\sigma$ is a two-dimensional smooth surface fibred into circles. Since $\sigma^{\text{hor}}(t) \in \mathcal{P}^a_{\sigma(t)}$ for any $t$ satisfies

$$\alpha^a(\sigma^{\text{hor}}(t), \dot{\sigma}^{\text{hor}}(t)) = 0 = \langle \sigma^{\text{hor}}(t) \times a(\sigma^{\text{hor}}(t)), \dot{\sigma}^{\text{hor}}(t) \rangle$$

and $\sigma^{\text{hor}}(t) \times a(\sigma^{\text{hor}}(t))$ is tangent to the circle $\mathcal{P}^a_{\sigma(t)}$, the curve $\sigma^{\text{hor}}$ intersects each fibre of $\mathcal{P}^a_{\sigma(t)}$ perpendicularly. Clearly, $\mathcal{P}^a|_\sigma$ is a rotational surface in $E$ provided the image of $\sigma$ is a straight line segment. Hence if $\mathcal{P}^a|_\sigma$ is a rotational surface each horizontal lift is a meridian.

In the case $\dot{\sigma}$ and $\ddot{\sigma}$ are linearly dependent, the horizontal lift of $\sigma$ to $\mathcal{P}^a|_\sigma$ is a pre-geodesic, i.e. a geodesic up to a reparametrization. This is the case if $\mathcal{P}^a|_\sigma$ is isometric to a rotational surface.

Obviously, if $\beta$ is an integral curve of the constant vector field and the solar field, respectively, a horizontal lift is a geodesic and a pre-geodesic on $\mathcal{P}^a|_\beta$, respectively, with the Riemannian metric given by the scalar product on $E$.

Next let us turn to a phenomenon which may arise if an integral curve $\beta$ of $X$ is closed. Here closed means that

$$\beta(t_0) = \beta(t_1)$$

for two values $t_0$ and $t_1$ in the domain of $\beta$. As is easily verified, a horizontal lift $\beta^{\text{hor}}$ satisfies

$$\beta^{\text{hor}}(t_1) = \beta^{\text{hor}}(t_0) \cdot z_\beta$$

for some $z_\beta \in U^a(1)$. This is to say $\beta^{\text{hor}}$ is not necessarily closed again. The complex number $z_\beta \in U^a(1)$ is called the *holonomy* of $\beta$. It can be computed as

$$z_\beta = \mathrm{e}^{-a\cdot\int_{S'}\Omega}$$

where $S'$ is a surface in $O$ bounded by $\beta$. (Clearly $a$ has to be replaced by i if $z_\beta$ is in $U(1)$.)

This concept of closedness at $x \in O$ can obviously be generalized to an arbitrary closed curve $\sigma$ in $O$. If $\mathcal{S}_x$ denotes the collection of all (smooth) curves in $O$ closed at $x \in O$, then

$$\{z_\sigma \mid \sigma \in \mathcal{S}_x\} \subset U^a(1)$$

forms a subgroup, called the *holonomy group* (cf Greub *et al* 1973). This subgroup is sensitive to the curvature.

The holonomy is interesting because of the following: Let us regard $\beta^{\text{hor}}(t)$ as a parallel transport of the initial internal variable $v_x$ along the trajectory $\beta$ with initial values $x$. In the case of a field line $\beta$ which closes in $x = \beta(t_1)$, the internal variable $\beta^{\text{hor}}(t_1)$ does not need to coincide with the initial value $v_x$ at all. A holonomy may affect this information transmission. If so, this transmission is affected by the curvature $\Omega$ caused by the vector field via $\mathcal{P}^a$.

Let $a$ be constant, now. A curve $\gamma$ on $\mathcal{P}^a|_\beta$ here is called a *periodic lift* of $\beta$ through $v_x$ iff it is of the form

$$\gamma(s) = \beta^{\text{hor}}_{v_x}(s) \cdot \mathrm{e}^{p\cdot s\cdot\frac{a}{|a|}} \in \mathcal{P}^a_{\beta(s)} \qquad \forall s \tag{6}$$

where $p$ is a fixed real.

Clearly, $\gamma$ is a horizontal lift through $v_x$ iff $\gamma = \beta^{\mathrm{hor}}_{v_x}$, i.e. iff $p = 0$. In fact any periodic lift $\gamma$ of $\beta$ is a geodesic on $\mathcal{P}^a|_\beta$. Hence $\ddot{\gamma}$ is perpendicular to $\mathcal{P}^a|_\beta$. Due to the $U(1)$-symmetry of $\mathcal{P}^a|_\beta$, a geodesic $\sigma$ on $\mathcal{P}^a|_\beta$ is of the form

$$\sigma(s) = \beta^{\mathrm{hor}}_{v_x}(\theta \cdot s) \cdot \mathrm{e}^{p\cdot\theta\cdot s\cdot \frac{a}{|a|}} \qquad \forall s$$

as is easily verified. Here $p$ and $\theta$ denote reals. $\theta$ determines the speed of the geodesic. Thus $\sigma$ and $\gamma$ have accordant speeds if $\theta = 1$, as can be easily seen from

$$\dot{\gamma}(s) = \left(p \cdot \frac{a}{|a|} \cdot \beta^{\mathrm{hor}}_{v_x}(s) + \dot{\beta}^{\mathrm{hor}}_{v_x}(s)\right) \cdot \mathrm{e}^{p\cdot s\cdot \frac{a}{|a|}} \qquad \forall s$$

and in particular from

$$\dot{\gamma}(0) = p \cdot v_x \cdot \frac{a}{|a|} + \dot{\beta}^{\mathrm{hor}}_{v_x}(0).$$

The real number $p$ determines the spatial frequency of the periodic lift $\gamma$ due to $\frac{2\cdot\pi}{T} = \frac{p}{|v_x|}$. Here $T$ is the period of revolution of a point on $\gamma$. Interpreting $s$ as a time, the spatial frequency of $\gamma$ counts the number of revolutions of a point on $\gamma$ around $\mathcal{P}^a|_\beta$ per unit time and is determined by the $F^{a(x)}$-component of the initial velocity. On the other hand $p$ is a first integral of the motion $\gamma$ due to the $U(1)$-symmetry of the cylinder $\mathcal{P}^a|_\beta$.

In the case of the non-constant vector field $X(x) = \left(x, \frac{x}{|x|^3}\right)$ with $x \in O$, i.e. the solar field with mass one, an integral curve can be parametrized by

$$\beta(t) = x_0 \cdot (3t-2)^{\frac{1}{3}} \quad \text{with} \quad \beta(1) = x_0.$$

Let $|x_0| = 1$ and let a parametrization of the body of revolution $\mathcal{P}^a|_\beta$ be given in Clairaut coordinates via the map $\mathbf{x} : \mathcal{U} \to E$ defined by

$$\mathbf{x}(u, v) := (3v-2)^{\frac{1}{3}} \cdot r\left(\mathrm{e}^{u\cdot\frac{a}{|a|}}\right) \cdot \left(v_{x_0} + \frac{a}{|a|}\right)$$

on an open set $\mathcal{U} \subset \mathbb{R}^2$. Here $r$ is the representation of $U^{\frac{a}{|a|}}(1)$ onto $SO\left(F^{\frac{a}{|a|}}\right)$ for any $x \in O$. Then a geodesic $\gamma$ on $\mathcal{P}^a|_\beta$ takes the form

$$\gamma(s) = \mathbf{x}(u(s), v(s)) = (3v(s)-2)^{\frac{1}{3}} \cdot r\left(\mathrm{e}^{u(s)\cdot\frac{a}{|a|}}\right) \cdot \left(v_{x_0} + \frac{a}{|a|}\right) \tag{7}$$

where the functions $u$ and $v$ are determined by

$$u(s) = \sqrt{2} \cdot \arctan\left(\frac{s}{\sqrt{2}c} + \frac{c_1}{2c}\right) + c_2 \tag{8}$$

and

$$v(s) = \pm\frac{1}{3}\left(\left(\frac{1}{\sqrt{2}}s + c_1\right)^2 + c^2\right)^{\frac{3}{2}} + \frac{2}{3} \tag{9}$$

(cf Pods 2003). Here $c_1$ and $c_2$ are integration constants determining the initial conditions. Since we are concerned with a forward movement, i.e. a transmission of information along the channel $\mathbb{R} \cdot \frac{a}{|a|}$, only the positive sign in (9) is of interest in $v(s)$. The constant $c$ fixes the slope of the geodesic via

$$\cos\alpha = \frac{c}{\sqrt{\left(\frac{1}{\sqrt{2}}s + c_1\right)^2 + c^2}}$$

where $\alpha$ is the constant angle between the geodesic $\gamma$, called periodic lift, again, and the parallels given in Clairaut coordinates. This means that $c$ vanishes precisely for a meridian.

A geodesic $\gamma$ is a horizontal lift of $\beta$ iff $\gamma$ is a meridian. Thus the parametrization of a meridian as a horizontal lift $\beta^{\mathrm{hor}}$ of an integral curve $\beta$ has the form

$$\beta^{\mathrm{hor}}(t) = v_x \cdot (3t-2)^{\frac{1}{3}}$$

with $\beta^{\mathrm{hor}}(1) = v_{x_0}$ as well as $\beta(1) = x_0$ for $\frac{2}{3} \leqslant t$ for any initial $v_x \in \mathcal{P}^a_{\beta(1)}$.

For the constant vector field from above, any periodic lift $\gamma$ of $\beta$ through $v_x$ is uniquely determined by the $U^a(1)$-valued map

$$s \mapsto \mathrm{e}^{p\cdot s\cdot \frac{a}{|a|}}$$

while for the solar field a periodic lift is characterized by

$$s \mapsto \mathrm{e}^{u(s)\cdot \frac{a}{|a|}}$$

with $u(s)$ as in (8). These two maps here are called an elementary periodic function, respectively, an elementary Clairaut map. Therefore, we can state:

**Proposition 2.** *Let $x$ be a point on the integral curve $\beta$. Under the hypothesis $a = const$ made above there is a one-to-one correspondence between all elementary periodic $U^a(1)$-valued functions and all periodic lifts of $\beta$ passing through a given $v_x \in \mathcal{P}^a_x$. In the case $X$ is the solar field, there is a one-to-one correspondence between the collection of all periodic lifts passing through a given $v_x \in \mathcal{P}^a$ and elementary Clairaut maps.*

Hence the periodic lifts of $\beta$ on $\mathcal{P}^a|_\beta$ describe the transmission of pieces of information of $\mathcal{P}^a|_\beta$ along $\beta$, the channel of information transmission.

Let us conclude this section with an interpretation of the internal variables of a constant vector field $X$ with principal part $a$. To this end we put the emphasis on a periodic lift on the cylinder $\mathcal{P}^a\big|_{\beta_{x_0}}$ for a given initial point $x_0$ of the integral curve $\beta_{x_0}$. This lift shall now be rotating with frequency $\nu$. Thus a point $w(\zeta;t)$, say, on this rotating lift is described by

$$w(\zeta;t) = \left|v_{x_0}\right| \cdot \frac{\beta^{\mathrm{hor}}_{v_{x_0}}(\zeta)}{\left|\beta^{\mathrm{hor}}_{v_{x_0}}(\zeta)\right|} \cdot \mathrm{e}^{-\nu(t-|a|\cdot p\cdot\zeta)\cdot\frac{a}{|a|}} \tag{10}$$

for all $\zeta \in \dot{\mathbb{R}}$ and any $t \in \mathbb{R}$. Thus $w$ is a circular polarized wave on the cylinder with $\frac{1}{|a|\cdot|p|}$ as speed of the phase and $\left|v_{x_0}\right|$ as amplitude; $w$ travels along $\mathbb{R}\cdot\frac{a}{|a|}$, the channel of information.

Thus we have associated a circular polarized wave of internal variables with the integral curve.

The map $w$ is a wave with values in the cylinder $\mathcal{P}^a\big|_{\beta_{x_0}}$, a flat manifold. It can be extended from $\mathbb{R}\cdot\frac{a}{|a|}$ to all of $E$ and be interpreted as an electric or magnetic field as will be seen in the next section.

## 5. Circular polarized electromagnetic waves of internal variables

Here we will explore a particular type of circular polarized wave, namely electromagnetic circular polarized plane waves. In the case of a constant vector field on $E$ with principal part having value $a$, internal variables may be interpreted as electric and magnetic field strengths, respectively.

To demonstrate this let us study the geodesics on $\mathcal{P}^a\big|_{\beta_x}$ where $a$ is constant for any $x \in E$. Given an initial point $x \in E$ and an initial vector $v_x \in \mathcal{P}^a_x$, the integral curve $\beta_x$ passing through $x$ at $\zeta = 0$ is covered by a geodesic $\gamma$ on $\mathcal{P}^a\big|_{\beta_x}$ through $v_x$. In fact, $\gamma$ passing through

$v_x$ can be viewed as a periodic lift of $\beta_x$ formed with respect to the Riemannian metric on $\mathcal{P}^a$ inherited from $E \times \mathbb{H}$. As mentioned in (6), the equation of $\gamma$ is

$$\gamma(\zeta) = \beta_x^{\mathrm{hor}}(\zeta) \cdot \mathrm{e}^{|a| \cdot p \cdot \zeta \cdot \frac{a}{|a|}} \qquad \forall \zeta \in \mathbb{R}$$

where $\beta_x^{\mathrm{hor}}$ is a meridian through $v_x$ at $\zeta = 0$ and $p$ is a fixed real. Moreover, $\zeta$ varies on the $\zeta$-axis $\mathbb{R} \cdot \frac{a}{|a|}$. Let the $\xi$-axis and the $\eta$-axis in $F^a$ be given by $\mathbb{R} \cdot \frac{v_x}{|v_x|}$ and $\mathbb{R} \cdot \frac{v_x}{|v_x|} \cdot \frac{a}{|a|}$, respectively.

The form of the equation of $\gamma$ suggests we consider the electric field

$$\mathbb{E} : E \times \mathbb{R} \longrightarrow \mathcal{P}^a \subset E \times E$$

defined by

$$\mathbb{E}(\xi, \eta, \zeta, t) := \frac{A}{|v_x|} \cdot v_x \cdot \mathrm{e}^{-\nu(t - \frac{\zeta}{c}) \cdot \frac{a}{|a|}} \qquad \forall \xi, \eta, \zeta, t \in \mathbb{R}$$

for any $x \in E$ with $x = (\xi, \eta, \zeta)$. Here $A$ is an amplitude, $\nu$ the frequency with which the geodesic as a whole rotates, $c$ the speed of light, $|a| = \frac{1}{c}$ and $p = 1$. Hence the electric field along any $\beta_x$ is caused by a rotation of a geodesic on the cylinder $\mathcal{P}^a\big|_{\beta_x}$ (cf (10)). Thus if $A = |v_x|$, for any $x \in E$ the values of the field $\mathbb{E}$ are internal variables. The vector

$$\mathbb{B}(\xi, \eta, \zeta, t) := \mathbb{E}(\xi, \eta, \zeta, t) \times \frac{a}{|a|} \qquad \forall \xi, \eta, \zeta, t \in \mathbb{R}$$

is the magnetic field vector.

It is a straightforward matter to show that $\mathbb{E}$ and $\mathbb{B}$ satisfy the Maxwell equations of non-conducting isotropic media, namely rot $\mathbb{B} = \frac{1}{c} \cdot \frac{\partial \mathbb{E}}{\partial t}$ and rot $\mathbb{E} = -\frac{1}{c} \frac{\partial \mathbb{B}}{\partial t}$ where $\varepsilon = \mu = 1$, and, in turn, the wave equations

$$\Delta \mathbb{E} = -\frac{1}{c^2} \frac{\partial^2 \mathbb{E}}{\partial t^2} \qquad \text{and} \qquad \Delta \mathbb{B} = -\frac{1}{c^2} \frac{\partial \mathbb{B}}{\partial t^2}$$

(cf Born 1985).

Hence internal variables of the constant vector field may be viewed as electric and magnetic field vectors. If the initial conditions of the electromagnetic waves (i.e. the values of a section into $\mathcal{P}^a$) are interpreted as pieces of information, these pieces of information are transmitted by a wave.

## 6. Heisenberg groups and algebras of the constant field and of the solar field

In the case of a constant vector field $X = (\mathrm{id}, a)$ the fibre $\mathcal{P}^a_{\beta_x(s)}$ is a circle with radius $|a|^{-\frac{1}{2}}$ and $\mathcal{P}^a|_{\beta_x}$ is a cylinder. The Heisenberg group bundle $\mathbb{G}^a$ introduced in section 2 is trivial, i.e.

$$\mathbb{G}^a = E \times G^a.$$

Each Heisenberg group $G^a$ induces a Heisenberg algebra

$$\mathcal{G}^a := \mathbb{R} \cdot a \oplus F^a$$

with Lie bracket

$$[\lambda_1 \cdot a + h_1, \lambda_2 \cdot a + h_2]_{\mathcal{G}^a} := \omega^a(h_1, h_2) \cdot a$$

for $\omega^a$ as defined in (2). The Heisenberg algebra bundle

$$\tilde{\mathcal{G}}^a := E \times \mathcal{G}^a$$

is trivial, as well.

For the radial vector field on $\dot{E} = E\backslash\{0\}$, i.e. for the solar field given by

$$a(x) = \frac{\bar{m}}{|x|^3} \cdot x \qquad \forall x \in \dot{E}$$

the fibre of the principal bundle $\mathcal{P}^a$ at $x$ is a circle with radius $\frac{|x|}{\sqrt{\bar{m}}}$. This shows that $\mathcal{P}^a|_{\mathbb{R}\cdot x}$ is a cone (without its vertex). For reasons of simplicity we assume $\bar{m} = 1$. The collection

$$\bigcup_{x\in S^2} U^{a(x)}(1) = S^3 \subset \mathbb{H}$$

is a group under the multiplication in $\mathbb{H}$. It is the group $SU(2)$. Moreover,

$$\mathbb{H} = \bigcup_{x\in\mathbb{R}\cdot\frac{x_0}{|x_0|}} S^3_{|a(x)|}$$

for a fixed $x_0 \in S^2$. Here $S^3_{|a(x)|}$ is the three-sphere with radius $|a(x)|$.

Next we will see that the group structure on $SU(2)$ is determined entirely by one fibre of $\mathbb{G}^a$, i.e. by one Heisenberg group. In doing so we follow Pods (2003). Observe that any fibre $G^{a(x)}$ is of the form

$$G^{a(x)} = \tau_{u(x)}(G^{a(x_0)})$$

for a given $x_0 \in S^2$. Here we again use the inner automorphism $\tau_h$ given by $\tau_h(k) := h\cdot k\cdot h^{-1}$ for all $k \in \mathbb{H}$ and any $h \in \mathbb{H}$ (cf (4)). (It is well known that any automorphism of the skew field $\mathbb{H}$ is an inner automorphism.) In analogy to the remarks in section 3, $a(x) = \tau_{u(x)}(a(x_0))$ for all $x \in S^2$. Therefore we can deduce all (three-dimensional) Heisenberg groups from a single group by applying a suitable inner automorphism. (This has in fact a whole variety of consequences.)

To determine the group structure of $SU(2)$ let us consider

$$K := \bigcup_{b\in S^2} G^b$$

containing $SU(2)$.

The group structure of $SU(2)$ can be reconstructed from the Heisenberg groups in $K$ as follows: As a set, $SU(2)$ is the union of the centres of the Heisenberg groups. Any two elements $w_1, w_2 \in SU(2)$ can be written in the form

$$w_1 = \cos t_1 \cdot e + \sin t_1 \cdot b_1 \qquad \in U^{b_1}(1) \qquad w_2 = \cos t_2 \cdot e + \sin t_2 \cdot b_2 \qquad \in U^{b_2}(1)$$

for suitable $b_1, b_2 \in S^2$. Then for the product $w_1 \cdot w_2$ we have

$$\begin{aligned} w_1 \cdot w_2 &= (\cos t_1 \cdot e + \sin t_1 \cdot b_1) \cdot (\cos t_2 \cdot e + \sin t_2 \cdot b_2) \\ &= \cos t_1 \cos t_2 \cdot e + \sin t_1 \cos t_2 \cdot b_1 + \cos t_1 \sin t_2 \cdot b_2 + \sin t_1 \sin t_2 \cdot b_1 \cdot b_2 \end{aligned}$$

so that $|w_1 \cdot w_2|^2 = 1$. Moreover, $b_1 \cdot b_2$ can be expressed by data in $K$. It can be shown that the multiplication defined in this way corresponds to the multiplication in $SU(2)$ given by the one in $\mathbb{H}$.

Vice versa, the multiplication in $\mathbb{H}$ and hence the one in $SU(2)$ determine the Lie structure on $\mathcal{G}^b$ for any $b \in S^2$ since the product of any two $h_1, h_2 \in F^b$ is given by equation (1) and

$$\langle h_1 \cdot h_2, b\rangle \cdot b = \langle h_1 \times h_2, b\rangle \cdot b = -\omega^b(h_1, h_2) \cdot b = -[h_1, h_2]_{\mathcal{G}^b}$$

holds.

## 7. Geometric construction of the Schrödinger representation

In this section we confine ourselves again to a constant vector field and to the solar field, starting with a constant vector field.

Unless specified otherwise, $\beta$ denotes a field line of $X$, a straight line, with initial condition $\beta(0) = x$. Again $\beta^{\text{hor}}_{v_x}$ is the horizontal lift of $\beta$ through $v_x \in \mathcal{P}^a_x$. There is a unique periodic lift $\gamma$ of $\beta$ passing through $v_x = \gamma(0)$ with prescribed velocity $\dot{\gamma}(0)$. First we will associate with $\dot{\gamma}(0)$ a well-defined unitary linear operator on a Hilbert space as follows:

The specification of $v_x \in \mathcal{P}^a_x$ turns $F^{a(x)}$ into a field isomorphic to $\mathbb{C}$, since $\frac{v_x}{|v_x|} \cdot \mathbb{C} = F^{a(x)}$. The real axis is $\mathbb{R} \cdot \frac{v_x}{|v_x|}$ and the imaginary one is $\mathbb{R} \cdot \frac{v_x}{|v_x|} \times \frac{a}{|a|}$. We rename these axes as the $q$-axis carried by the unit vector $\bar{q}_x$ and the $p$-axis carried by the unit vector $\bar{p}_x$, respectively. Any $h \in F^{a(x)}$ is thus of the form $h = (q, p)$. The Schwartz space of the real axis and its $L^2$-completion, a Hilbert space, are denoted by $\mathcal{S}(\mathbb{R}, \mathbb{C})$ and $L^2(\mathbb{R}, \mathbb{C})$, respectively.

Given a frequency $\nu$, the Schrödinger representation $\rho^\nu_x$ of $G^{a(x)}$ (containing $F^{a(x)}$) acts on each complex-valued $\psi \in \mathcal{S}(\mathbb{R}, \mathbb{C}) \subset L^2(\mathbb{R}, \mathbb{C})$ by

$$\rho^\nu_x(z + h)(\psi(\tau)) := \mathrm{e}^{-\nu\cdot\vartheta\cdot\frac{a}{|a|}} \cdot \mathrm{e}^{|a|\cdot\nu\cdot p\cdot\tau\cdot\mathrm{i}} \cdot \mathrm{e}^{-\frac{|a|}{2}\cdot\nu\cdot p\cdot q\cdot\mathrm{i}} \cdot \psi(\tau - q) \tag{11}$$

for all $\tau \in \mathbb{R}$ and all $z + h = \mathrm{e}^{\vartheta\cdot\frac{a}{|a|}} + (q, p) \in G^{a(x)}$ for some $\vartheta \in \mathbb{R}$ (cf Schempp 1986, Folland 1989, Guillemin and Sternberg 1991). Clearly,

$$-p \cdot q \cdot \mathrm{i} = \omega^a((p, 0), (0, q)) \cdot \mathrm{i}.$$

By the Stone–von Neumann theorem $\rho^\nu_x$ is irreducible (cf Schempp 1986, Folland 1989).

Setting $\nu = 1$, $\rho^\nu_x = \rho$, $z = e$ and $q = |v_x|$, for any $p \in \mathbb{R}$ equation (11) turns into

$$\rho(e + (|v_x|, p))(\psi)\left(\tau + \frac{|v_x|}{2}\right) = \mathrm{e}^{|a|\cdot p\cdot\tau\cdot\mathrm{i}} \cdot \psi\left(\tau - \frac{|v_x|}{2}\right) \qquad \forall \tau \in \mathbb{R}.$$

Operators of this form generate $\rho(G^{a(x)})$, of course.

In the case the frequency $\nu$ differs from one, for each $p \in \mathbb{R}$ equation (11) becomes

$$\rho^\nu_x\big(\mathrm{e}^{t\cdot\frac{a}{|a|}} + (|v_x|, p)\big)(\psi)\left(\tau + \frac{|v_x|}{2}\right) = \mathrm{e}^{-\nu\cdot(t-|a|\cdot p\cdot\tau)\cdot\mathrm{i}} \cdot \psi\left(\tau - \frac{|v_x|}{2}\right) \qquad \forall \tau, t \in \mathbb{R}. \tag{12}$$

This shows that the factor $\mathrm{e}^{-\nu(t-|a|\cdot p\cdot s)}$ for $s = \tau$, characteristic of a circular polarized wave as described in (12), shows up in the Schrödinger representation. This expresses the fact that the geometry of the collection $\mathcal{P}^a\big|_{\beta_{x_0}}$ of all internal variables along $\beta_{x_0}$ is directly transferred to the Hilbert space $L^2(\mathbb{R}, \mathbb{C})$ via the Schrödinger representation. Differently formulated, the Schrödinger representation has a geometric counterpart, namely $\mathcal{P}^a$ together with its geometry, for which the elements are bits of information.

On the other hand the $U^a_x(1)$-valued function $\tau \longrightarrow \mathrm{e}^{-\nu(t-|a|\cdot p\cdot\tau)\cdot\mathrm{i}}$ entirely describes the periodic lift $\gamma$, rotating with frequency $\nu$ and passing through $v_x$, as expressed in (10). Thus the circular polarized wave $w$ is characterized by the unitary linear transformation $\rho^\nu_x\big(\mathrm{e}^{t\frac{a}{|a|}} + (|v_x|, p)\big)$ on $L^2(\mathbb{R}, \mathbb{C})$. Due to the Stone–von Neumann theorem, the equivalence class of $\rho^\nu_x$ is uniquely determined by $\nu$ and vice versa.

Therefore, we state:

**Theorem 3.** *Any periodic lift $\gamma$ of $\beta$ on $\mathcal{P}^a|_\beta$ with initial conditions $\gamma(0) = v_x$ and first integral $p$ is uniquely characterized by the unitary linear transformation $\rho(e + (|v_x|, p))$ of $L^2(\mathbb{R}, \mathbb{C})$ with $(e + (|v_x|, p)) \in G^{a(x)}$ and vice versa. The unitary linear transformation $\rho^\nu_x\big(\mathrm{e}^{t\cdot\frac{a}{|a|}} + (|v_x|, p)\big)$ of $L^2(\mathbb{R}, \mathbb{C})$ characterizes the circular polarized wave $w$ with frequency $\nu$ on $\mathcal{P}^a|_\beta$ generated by $\gamma$ and vice versa. Thus $v_x \in \mathcal{P}^a_x$ determines a unitary representation*

*$\rho$ on $L^2(\mathbb{R}, \mathbb{C})$ characterizing the collection $C^a_{v_x}$ of all periodic lifts of $\beta$ passing through $v_x$, and, in turn, the circular polarized waves (in the sense above) with a given frequency $\nu$. The frequency determines the equivalence class of $\rho^\nu_x$ and vice versa.*

We may reinterpret theorem 3 as follows: Any curve $\gamma \in C^a_{v_x}$ on $\mathcal{P}^a|_\beta$ with prescribed first integral $p$ is an evolution of the piece of information $v_x$, transmitted by $\gamma$ with a velocity determined by $p$ or, equivalently, by the circular polarized wave with frequency $\nu$. Hence $\rho(e + (|v_x|, p))$ describes this transmission of a piece of information along the field line $\beta$ of $X$ in terms of a unitary linear operator. Thus the information encoded in $F^a$ is modulated on any signal in $L^2(\mathbb{R}, \mathbb{C})$. Since these operators generate the representation of $G^{a(x)}$, we may state the following:

**Corollary 4.** *The Schrödinger representation of $G^{a(x)}$ describes the transmission of any piece of information $(|v_x|, p) \in T_{(|v_x|,0)}\mathcal{P}^a|_\beta$ along the field line $\beta$, with $\mathbb{R} \cdot a$ as information transmission channel. It modulates the bits of information on any signal in $L^2(\mathbb{R}, \mathbb{C})$.*

The mechanism which associates with each geodesic a Schrödinger representation as expressed in theorem 3 generalizes for the solar field as follows (cf Pods 2003): Let $O = \dot{E}$. Given an integral curve $\beta$, we consider the Heisenberg algebra $\mathbb{R} \cdot \frac{a}{|a|} \oplus F^a$ equipped with the symplectic structure determined by $\frac{a}{|a|}$. Now let $\gamma$ be a geodesic on the rotational surface $\mathcal{P}^a|_\beta$. The curve parameter is $s$ and the surface shall be parametrized by $\mathbf{x}(u, v)$ with coordinate functions $u$ and $v$ (cf (7)). Thus $\gamma(s)$ is identical with the point $\mathbf{x}(u(s), v(s))$ on the surface. In addition, let $\psi \in \mathcal{S}(\mathbb{R}, \mathbb{C})$. Then the Schrödinger representation $\rho_s$ (with frequency one) on the Heisenberg group $G^{a(x)}$ of the solar field is given by

$$\rho_s(z, \mathbf{x}(u(s), v(s)))(\psi)(\tau) := z^{-1} \cdot \mathrm{e}^{u(s)\cdot\tau\cdot\mathrm{i}} \cdot \mathrm{e}^{\frac{1}{2}\cdot u(s)\cdot v(s)} \cdot \psi(\tau - v(s))$$

for all $s$ in the domain of $\gamma$ and any $\tau \in \mathbb{R}$. Obviously this further generalizes to any frequency.

## 8. Spin $\frac{1}{2}$ and Schrödinger representations

The map $\tau : SU(2) \to SO(E)$ assigns to any $h$ the isometry $\tau_h|_E$. Moreover, $\tau$ is surjective and two-to-one. Hence $SU(2)$ is the spin group in the Clifford algebra $\mathbb{H}$ of $F^b$ equipped with $-\langle\, , \rangle$. Since $\tau_{t\cdot h} = \tau_h$ for any $h \in SU(2)$ and any $t \in \mathbb{R}$, the group $SO(E)$ is the real projective space of $\mathbb{H}$, a fact which has remarkable consequences in quantum mechanics and MRI (cf Schempp 1998).

Since the group structure of $SU(2)$ is reconstructable from the Heisenberg groups in $K = \cup_{b\in S^2} G^b$, one expects a close relationship between spin representations and Schrödinger representations of the Heisenberg groups in $K$. This relationship, established in Pods (2003), is fundamental for signal theory, in particular for MRI. It allows us to describe signal transmission and detection in three-space with spin models.

We here concentrate on the spin $\frac{1}{2}$ representation of $SU(2)$ only.

Given a spin $\frac{1}{2}$ representation $r$, the representation space has complex dimension two and real dimension four, respectively. Thus we may take $\mathbb{H} = \mathbb{C}^a \oplus F^a$, a complete unitary complex linear space, as representation space.

Fixing some $b \in S^2$, the restriction $r^b$ of $r$ to $U^b(1)$ is not irreducible; in fact, it splits according to $\mathbb{H} = \mathbb{C}^b \oplus F^b$ into $r^b = r^b_1 + r^b_2$. The (unitary) first component

$$r^b_1 : U^b(1) \to U(\mathbb{C}^b)$$

acts as

$$r^b_1(z) = m_z \qquad \forall z \in U^b(1)$$

while the (unitary) second one

$$r_2^b : U^b(1) \to U(F^b)$$

operates via

$$r_2^b(z) = m_{z^{-1}} \qquad \forall z \in U^b(1).$$

Here $m_z$ and $m_{z^{-1}}$ denote the right multiplications by $z$ on $\mathbb{C}^b$ and by $z^{-1}$ on $F^b$, respectively.

Elements of the form

$$k := \tfrac{1}{2} r(a)(h) \qquad \forall h \in SU(2)$$

are called spinors.

On the other hand, the Schrödinger representation on $G^b$, say, is based on the following splitting of the Heisenberg group. Given $v \in F^b$ and $b \in S^2$, let

$$K^b := \mathbb{R} \cdot v \cdot b \qquad \text{and} \qquad N^b := U^b(1) \oplus \mathbb{R} \cdot v.$$

Then

$$G^b = K^b \oplus N^b.$$

The character

$$\chi_{r_s}^b := \chi_{r_s} \cdot \mathrm{e}^{\langle v, \ldots \rangle \cdot \mathrm{i}} \qquad \text{for}\, s = 1, 2$$

on $G^b$, where $\chi_{r_s} = \operatorname{tr} r_s$ for $s = 1, 2$ and $\mathrm{e}^{\langle v, \ldots \rangle \cdot \mathrm{i}}$ is the character on the Abelian group $F^b$ mapping any $w \in F^b$ into $\mathrm{e}^{\langle v, w \rangle \cdot \mathrm{i}}$, yields, after an induction, the Schrödinger representation $\rho_s^b$ for $s = 1, 2$ with frequency 1 and $-1$, respectively. Different $v \in F^b \backslash \{0\}$ yield equivalent representations due to the Stone–von Neumann theorem and the Kirillov coadjoint orbit theorem for nilpotent Lie groups (cf Schempp 1986).

Thus

$$\rho_r^b := \rho_1^b + \rho_2^b$$

is a representation of $G^b$ onto the unitary group $U(L^2(\mathbb{R}, \mathbb{C}) \oplus L^2(\mathbb{R}, \mathbb{C}))$. In the case $\nu = 0$, both $\rho_1^b$ and $\rho_2^b$ are one-dimensional. Hence, if $Y$ is a (not necessarily continuous) vector field on $S^2$, the pair $(r, Y)$ provides a family $\rho_Y$ of Schrödinger representations emanating from $K$. The representations $\rho_Y$ and $\rho_{Y'}$ will be equivalent as long as they have the same singularities. Vice versa, let $\rho_1^b$ and $\rho_2^b$ be two Schrödinger representations of the Heisenberg group $G^b$ with $b \in S^2$ with respective frequencies 1 and $-1$. They extend to any $U^{b'}(1) \subset SU(2)$ with $b' \in S^2$ by setting

$$\rho_1^{b'} := \rho_1^b \circ \tau_k \qquad \text{and} \qquad \rho_2^{b'} := \rho_2^b \circ \tau_k$$

for some $k \in \mathbb{H}$ with $\tau_k(b') = b$. They determine a character $\chi$ on $SU(2)$ given for any $t \in \mathbb{R}$ by

$$\chi(\mathrm{e}^{t \cdot b'}) = \exp(-\mathrm{i} \cdot t) + \exp(\mathrm{i} \cdot t) \qquad \forall b' \in S^2 \subset E = su(2).$$

Hence $\chi$ determines a spin $\frac{1}{2}$ representation of $SU(2)$ which reproduces $\rho_1^b$ and $\rho_2^b$ by means of the above construction. There is an infinitesimal version of this link, as well (cf Pods 2003).

The relation between the spin $\frac{1}{2}$ representations and the Schrödinger representations as mentioned above emanates from *one* Heisenberg algebra $\mathcal{G}^b$ with $b \in S^2$.

Given one of the Schrödinger representations $\rho_1^b$ and $\rho_2^b$ of the Heisenberg group $G^b$ (the other one can be reconstructed), there is a spin $\frac{1}{2}$ representation. Moreover, given one Heisenberg algebra $\mathcal{G}^b$ and an (infinitesimal) Schrödinger representation with frequency one, there comes a (four-dimensional) spin $\frac{1}{2}$ representation of $SU(2)$ with it.

Let us point out that $\rho_1^b$ and $\rho_2^b$ are representations of the Heisenberg groups $(U^b(1) \oplus F^b, \omega^b)$ and $(U^{-b}(1) \oplus F^{-b}, \omega^{-b})$, respectively. In other words, $\rho_2^b$ is the contragradient representation of $\rho_1^b$. This is particularly important because $\rho_1^b \otimes \rho_2^b$ (emanating from the spin $\frac{1}{2}$ representation) is the principal ingredient to describe entangled pairs (photons etc), i.e. teleportation (cf Binz and Schempp 1999). Hence an equivalent way of saying this is that the spin $\frac{1}{2}$ representation encodes entangled pairs.

To deal with the notion of spin later on, let us discuss

$$e, -b, q_0, q_0 \cdot b$$

which is the skew analogue of Pauli matrices corresponding to the splitting $\mathbb{H} = \mathbb{C}^b \oplus F^b$ since $\mathbb{C}^b$ acts from the right on $F^b$ (cf Binz *et al* 2003).

Setting $z = \mathrm{e}^{t \cdot b}$ yields

$$\frac{b}{2} \cdot \dot{r}(b)(\mathrm{e}) = -\frac{1}{2} \cdot \mathrm{e} \qquad \frac{b}{2} \cdot \dot{r}(b)(b) = -\frac{1}{2} \cdot b$$
$$\frac{b}{2} \cdot \dot{r}(b)(q_0) = \frac{1}{2} \cdot q_0 \qquad \text{and} \qquad \frac{b}{2} \cdot \dot{r}(b)(q_0 \cdot b) = \frac{1}{2} \cdot q_0 \cdot b.$$

$\frac{1}{2} \cdot b \cdot \dot{r}$ is the classical (Hermitian) spin $\frac{1}{2}$ representation, while $\frac{\hbar}{2} \cdot b \cdot \dot{r}$ is the quantum mechanical (Hermitian) one. The eigenvalues are $\pm\frac{1}{2}$ and $\pm\frac{\hbar}{2}$, respectively.

Hence, if there is specified one Heisenberg algebra $\mathcal{G}^b$ or one Heisenberg group $G^b$, there is a classical and a quantum spin $\frac{1}{2}$ formalism present which allow the rephrasing of the information encoded in the Schrödinger representation (with frequency one) of $G^b$.

For MRI it is important to transfer the notion of spin to a point in the coadjoint orbits of $G^b$, pulled back to the Heisenberg algebra via the scalar product. These orbits are of the form

$$\vartheta \cdot b \oplus F^b \qquad \text{for a fixed } \vartheta \in \mathbb{R}.$$

Each of these affine planes represents a slice through the body. These slices are to be imaged. Each orbit is obtained by a translation $S_{\vartheta \cdot b}$, say, on the Euclidean space $E$, applied to $F^b$, i.e. $\vartheta \cdot b + F^b = S_{\vartheta \cdot b}(F^b)$. The half-spin

$$s = \frac{b}{2} \cdot \dot{r}(b)(h) \in T_0\mathcal{G}^b = \mathcal{G}^b$$

transferred to any point $\vartheta \cdot b + q$ in $\vartheta \cdot b \oplus F^b$ is

$$T_q S_\vartheta(s) = s \in T_{\vartheta \cdot b + q}(\mathbb{R} \cdot b + F^a)$$

since $T_q S_{\vartheta \cdot b} = \mathrm{id}$. In more detail this is discussed in Pods (2003).

The notion of spin $\frac{1}{2}$ can immediately be related to a magnetic field $B$, a two-form on $O$, since

$$B(x)(v, w) = \langle \mathbb{B}(x) \times v, w \rangle \qquad \forall v, w \in E \quad \text{and} \quad \forall x \in O.$$

For our purposes we suppose that the magnetic field strength $\mathbb{B}$ is constant on $O$. Thus any vector in $E = su(2)$ couples to $\mathbb{B}$. The constant field $\mathbb{B}$ defines a Heisenberg algebra $\mathcal{G}^{\mathbb{B}}$ along a field line, hence a Schrödinger representation of $\mathcal{G}^{\mathbb{B}}$ and, in turn, a spin $\frac{1}{2}$ representation of $SU(2)$. Given a spin $s = \dot{r}(b)(h)$ with $h \in \mathbb{H}$, the magnetic moment $\mu_s$ is defined to be

$$\mu_s := \mu \cdot s$$

where $\mu$ is the gyromagnetic (coupling) constant. The equation describing the motion of $s$ is given by

$$\frac{\mathrm{d}\mu_s}{\mathrm{d}t} = \mu \cdot s \times \mathbb{B}$$

which in turn yields the Larmor frequency $\omega$ of resonance

$$\omega = \mu \cdot |\mathbb{B}|.$$

This relation between spin $\frac{1}{2}$ and $\mathbb{B}$ is fundamental in MRI and yields a description of the imaging technique via the Schrödinger representation. This is done in Schempp (1998) and Pods (2003).

## 9. The ambiguity function

Next let us bring the Schrödinger representation into the context of time-frequency analysis. One of the basic ingredients is the *ambiguity function*.

Given any two signals $\varphi$ and $\psi$ in the Schwartz space $\mathcal{S}(\mathbb{R}, \mathbb{C})$ of rapidly decreasing $\mathbb{C}$-valued functions on $\mathbb{R}$, the ambiguity function $\mathcal{A}$ is defined by

$$\mathcal{A}(\varphi, \psi; \alpha, \beta) := \int_{\mathbb{R}} \mathrm{e}^{t\cdot\beta\cdot\mathrm{i}} \cdot \varphi\left(t - \frac{\alpha}{2}\right) \cdot \bar{\psi}\left(t + \frac{\alpha}{2}\right) \mathrm{d}t.$$

Here the reals $\alpha$ and $\beta$ are called time shift and frequency shift, respectively. This map

$$\mathcal{A}(\ldots, \ldots; \alpha, \beta) : \mathcal{S}(\mathbb{R}, \mathbb{C}) \times \mathcal{S}(\mathbb{R}, \mathbb{C}) \longrightarrow \mathcal{S}(\mathbb{R} \times \mathbb{R}, \mathbb{C})$$

is surjective and sesquilinear for each fixed $\alpha$ and $\beta$. Given $b \in S^2$, the function $\mathcal{A}$ is obviously adapted to $F^b$ via a coordinate system given by a chosen unit vector $q_0$ and $q_0 \cdot b$.

The ambiguity function is directly related to the Schrödinger representation (cf Schempp 1986) in the following way. The coefficient function $c_{\rho,\varphi,\psi}$ of $\rho$ (with frequency one) is defined by

$$c_{\rho,\varphi,\psi}(h) := (\rho(h)(\varphi), \psi)_{L^2}$$

for any $h \in F^b$, where both $\varphi$ and $\psi$ are defined on $\mathbb{R} \cdot q_0$. Here $(,)_{L^2}$ is the usual $L^2$-Hermitian product on $\mathcal{S}(\mathbb{R}, \mathbb{C})$. Setting $h = \alpha \cdot q_0 + \beta \cdot q_0 \cdot b \in F^b$ for $\alpha, \beta \in \mathbb{R}$ and $b \in S^2$,

$$c_{\rho,\varphi,\psi}(\alpha, \beta) = \int \mathrm{e}^{p\cdot(q-\frac{\alpha}{2})\cdot\mathrm{i}} \cdot \varphi(q - \alpha) \cdot \bar{\psi}(q) \, \mathrm{d}q$$

which yields

$$c_{\rho,\varphi,\psi}(\alpha, \beta) = \mathcal{A}(\varphi, \psi; \alpha, \beta)$$

for $t := q - \frac{\alpha}{2}$. In fact, $\rho$ is square integrable modulo centre, i.e. on $F^b = G^b/_{U^b(1)}$.

Since the information encoded in the ambiguity function $\mathcal{A}$ is also hidden in $\rho$, we have a link between the Schrödinger representation, the spin $\frac{1}{2}$ representation and the ambiguity function, which may be used for a description of imaging techniques in MRI (cf Schempp 1998, Pods 2003). If, in particular, the spin $\frac{1}{2}$ formalism models microscopic processes, the derived Schrödinger representations $\rho_1$ and $\rho_2$ from the previous section make them detectable on the macroscopic scale (cf Pods 2003).

The well-known Wigner function is up to a rotation in the arguments of the Fourier transform of the ambiguity function (cf Gröchenig 2000). This function, however, was introduced by Wigner as a substitute for the non-existing joint probability distribution of position and momentum in the quantum state $\psi$. This hints at a link from the Schrödinger representation to quantum mechanics, which will become evident in the next section where we will discuss the quantization of inhomogeneous quadratic polynomials.

## 10. Quantization

The notion of quantization we look at here is based on the Heisenberg group $G^a$, the symplectic group $\mathrm{Sp}(F^a)$ and its two-fold covering $\mathrm{Mp}(F^a)$, the metaplectic group. Here $F^a \subset G^a$ is the symplectic plane for a given $a \in S^2$. The pointal information encoded in $F^a$ is modulated on signals via the Schrödinger representation. The points in $F^a$ will be rearranged by a symplectic map, i.e. by a linear map preserving the symplectic structure, which is nothing other than the volume form. This rearrangement of information inscribed in $F^a$ will yield a quantization. The following shall motivate the importance of volume preserving maps.

First of all, however, we will take a detour into information transmission in order to clarify the terms used in the following. Suppose we had a photography in a plane $F^a$, say, embedded into a Euclidean space $E$ and want to transmit the picture to a destination. For the sake of simplicity, the transmission will follow an axis $\mathbb{R} \cdot a$, where $a$ is transverse to the plane $F^a$. This axis shall be the channel of information.

Describing this process requires analysing the *information* the photography presents, the *geometry* underlying the *transmission*, the *encoding in signals* and their *detection* as well as the *preservation of information* during the transmission.

In this section we will mainly concentrate on the analysis of the notion of information, on the geometry of the transmission and, most important, on the preservation of the information during the transmission. The latter will directly yield the elementary quantization formalism, as we will see later.

First let us concentrate on the notion of information. We will enlarge our concept of information here from a pointal description to density maps, i.e. the information the photography contains is encoded in grey-scales expressed by a positive *density function*

$$f : F^a \longrightarrow \mathbb{R}.$$

A bit of information is a point $q \in F^a$ together with a grey value $f(q)$. Thus $f$ is here called an *information density*.

We define the total *amount $j$ of information* as

$$j := \int_F^a f \cdot \omega$$

where $\omega$ is a (non-vanishing) *volume form* on $F^a$. This form $\omega$ shall not depend on the points in $F^a$; it is a *symplectic form* expressing the grey-shade at each point $q \in F^a$. Its value is one if the point is a black spot and is close to zero if the spot is almost white.

Obviously,

$$\frac{f}{j} : F^a \to \mathbb{R}$$

is a probability density. Associated with it is one of the most basic ingredients of information theory, namely Shannon's *entropy*

$$\varepsilon := -\log \frac{f}{j}$$

(cf Cover and Thomas 1991). Roughly speaking, the entropy provides an upper bound for transmitting reliable information through a channel.

Next let us turn to the geometry underlying the information transmission. It is provided by the Heisenberg Lie algebra $\mathcal{G}^a = \mathbb{R} \cdot a \oplus F^a$ with symplectic structure $\omega$. The centre $\mathbb{R} \cdot a$ of $\mathcal{G}^a$ is our channel along which information shall be transmitted.

Again, the points in $F^a$ are regarded as internal variables of the points in $\mathbb{R} \cdot a$.

In general, the transmission of information follows much more general curves than straight lines, e.g., in geometric optics (cf Born 1985). However, the setting chosen is general enough to provide us with some of the most elementary observations on information transmission.

As mentioned above, the transmission of the information encoded in $F^a$ shall follow along $\mathbb{R} \cdot a$. In mathematical terms, the points in $F^a$ shall be smoothly mapped into $t \cdot a + F^a$ for any $t \in \mathbb{R}$. Due to our assumptions this map can be composed by a smooth map

$$\Phi(t) : F^a \to F^a$$

followed by a translation

$$\Phi_t : F^a \longrightarrow t \cdot a + F^a$$

where $\Phi_t(h) = t \cdot a + h$ for all $h \in F^a$. In contrast to $\Phi_t$, which is certainly information preserving, the map $\Phi(t)$ rearranges the information in $F^a$. If we wish the information to be preserved, $\Phi(t)$ has to be at least a diffeomorphism. Let us assume that $\Phi(0) = \mathrm{id}_{F^a}$. In order to investigate further requirements that guarantee information preservation we study $f(t) := f \circ \Phi(t)$ in the formula $j(t) := \int_{F^a} f(t) \cdot \omega(t)$ for the total information $j$ in $t \cdot a + F^a$ where $\omega(t)$ is the volume form on $F^a$ caused by the diffeomorphism $\Phi(t)$ on $F^a$. Clearly

$$\omega(t) = \Phi(t)^*\omega \qquad \forall t \in \mathbb{R}$$

which, expressed in more detail, reads

$$\omega(t)(v, w) = \det D\Phi(t) \cdot \omega(v, w)$$

for any $v, w \in F^a$ and any $t \in \mathbb{R}$. Thus the total information can be rewritten as

$$j(t) = \int_{F^a} f \circ \Phi(t) \cdot (\det D\Phi(t)) \cdot \omega \qquad \forall t \in \mathbb{R}. \tag{13}$$

The total information is preserved iff

$$\frac{\mathrm{d}j(t)}{\mathrm{d}t} = 0 \qquad \forall t \in \mathbb{R}.$$

The mean grey value expressed by the total information is not very informative, however. More informative is the requirement that the information at any pixel is preserved, which means that the integrand on the right-hand side of (13) is constant. This is to say

$$Df \circ \frac{\mathrm{d}\Phi(t)}{\mathrm{d}t} \cdot \Phi(t)^*\omega + f \circ \Phi(t) \frac{\mathrm{d}}{\mathrm{d}t}(\det D\Phi(t)) \cdot \omega = 0 \tag{14}$$

has to be satisfied for any $t \in \mathbb{R}$. This equation is called the *continuity equation* for $f$ (cf Binz 1993).

Since the information density $f(t)$ is positive for any $t \in \mathbb{R}$, the entropy

$$\varepsilon(t) = -\log \frac{f(t)}{j(t)}$$

is well defined. Thus equation (14) turns into

$$\frac{\mathrm{d}\varepsilon(t)}{\mathrm{d}t} = \mathrm{tr}\, D\Phi(t)^{-1} \circ \frac{\mathrm{d}}{\mathrm{d}t} D\Phi(t) \qquad \forall t \in \mathbb{R}.$$

We call the transmission *information preserving* iff the entropy is preserved. Since

$$\frac{\mathrm{d}f(t)}{\mathrm{d}t} = 0 \quad \Leftrightarrow \quad \frac{\mathrm{d}\varepsilon(t)}{\mathrm{d}t} = 0 \qquad \forall t \in \mathbb{R}$$

the information is preserved iff

$$\det D\Phi(t) = 1 \qquad \forall t \in \mathbb{R}$$

saying that

$$D\Phi(t) \in \mathrm{Sp}(F^a) \qquad \forall t \in \mathbb{R}.$$

We therefore may state

**Theorem 5.** *The information density $f$ is preserved iff the entropy is preserved which is equivalent to*

$$D\Phi(t) \in \mathrm{Sp}(F^a) \qquad \forall t \in \mathbb{R}.$$

This theorem is the basis for our set-up of quantization. To explain this we need the Schrödinger representation. We here essentially follow Guillemin and Sternberg (1991).

The Schrödinger representation $\rho^\nu$ (cf section 7) with frequency $\nu$ modulates the group $G^a$ onto each signal $\varphi \in \mathcal{S}(\mathbb{R}, \mathbb{C})$. This is to say that it modulates in particular the points of $F^a$ onto each signal $\varphi$. Referring to a point $h = q \cdot e_{\bar{q}} + p \cdot e_{\bar{p}}$ in $F^a$ as a bit of information, we can say that $\rho^\nu$ modulates information in $F^a$ onto any signal by

$$\rho^\nu(q \cdot e_{\bar{q}} + p \cdot e_{\bar{p}})(\varphi)(\tau) = \mathrm{e}^{-\nu \cdot \frac{p \cdot q}{2} \cdot \mathrm{i}} \cdot \mathrm{e}^{\nu \cdot p \cdot \tau \cdot \mathrm{i}} \cdot \varphi(\tau - q)$$

for any $\varphi \in \mathcal{S}(\mathbb{R}, \mathbb{C})$ and any $\tau \in \mathbb{R}$. Obviously, for any $\varphi \in \mathcal{S}(\mathbb{R}, \mathbb{C})$ and any $\tau \in \mathbb{R}$,

$$\rho^\nu(q \cdot e_{\bar{q}} + p \cdot e_{\bar{p}})(\varphi)(\tau) = \rho(q \cdot e_{\bar{q}} + \nu \cdot p \cdot e_{\bar{p}})(\psi)(\tau)$$

holds true. This means that as far as the modulation of information on signals from $\mathcal{S}(\mathbb{R}, \mathbb{C})$ is concerned, we can restrict ourselves to the Schrödinger representation $\rho$ having frequency $\nu = 1$.

To study the effect of a volume preserving rearrangement of the collection $\Sigma \subset F^a$ of information on a Schrödinger representation, we extend any $A \in \mathrm{Sp}(F^a)$ to all of $G^a$ by setting it equal to the identity on the centre $U^a(1)$. This extension is again called $A$. Hence

$$A : G^a \longrightarrow G^a$$

is a group automorphism. The infinitesimal transmission of information is a symplectic map on $F^a$, provided the information density is preserved, regardless of the specific nature of this density.

If the information is modulated on $\mathcal{S}(\mathbb{R}, \mathbb{C})$, the preservation of information is expressed by the Stone–von-Neumann theorem implying

$$\rho \circ A = U(A) \circ \rho \circ U(A^{-1})$$

for any $A \in \mathrm{Sp}(F^a)$. Here $U$ is the metaplectic representation. This expresses the fact that the preservation of information during the transmission is rephrased by maintaining the equivalence class of the Schrödinger representation during the transmission. In short, preserving the entropy of the information means preserving the equivalence class of the information.

This fact, however, yields classical quantum mechanics, i.e. the quantization of all inhomogeneous quadratic polynomials, as we will see next. Thus the preservation of entropy paired with the Schrödinger representation causes classical quantum mechanics.

Therefore we have a representation

$$U : \mathrm{Mp}(F^a) \longrightarrow U(L^2(\mathbb{R}, \mathbb{C}))$$

the *metaplectic representation*.

Hence a volume preserving rearrangement, i.e. an *isentropic* rearrangement, of information in $F^a$ causes the irreducible metaplectic representation. We will see below that $U$ and the Schrödinger representation $\rho$ yield a quantization procedure for inhomogeneous quadratic polynomials.

We will base the quantization scheme on the *infinitesimal metaplectic representation* and on $\mathrm{d}\rho$. The infinitesimal representation $\mathrm{d}U$ of $U$, i.e. the representation of the Lie algebra $\mathrm{mp}(F^a)$ is the differential of $U$ at $\mathrm{id}_{F^a} \in \mathrm{Mp}(U)$. Since the metaplectic group $\mathrm{Mp}(F^a)$ is a two-fold covering of the symplectic group $\mathrm{Sp}(F^a)$, both Lie algebras $\mathrm{sp}(F^a)$ and $\mathrm{mp}(F^a)$ are identical. Thus we base our developments on $\mathrm{sp}(F^a)$, which consists of all traceless linear endomorphisms of $F^a$.

Our first goal is to establish a natural isomorphism between $\mathrm{sp}(F^a)$ and the Poisson algebra of all *homogeneous quadratic polynomials* $\mathcal{Q}$. The latter will be introduced now.

We define a Poisson bracket on $C^\infty(F^a, \mathbb{R})$, the $\mathbb{R}$-algebra of all smooth $\mathbb{R}$-valued functions of $F^a$: given $f \in C^\infty(F^a, \mathbb{R})$, its Hamiltonian vector field $X_f$ is defined by

$$\omega^a(X, X_f) = \mathrm{d}f(X)$$

for any smooth vector field $X$ on $F^a$.

Clearly $X_f$ is smooth. For any $g$ in the collection $C^\infty(F^a, \mathbb{R})$ of all $\mathbb{R}$-valued smooth function of $F^a$, one easily observes that the Lie bracket $[X_f, X_g]$ is a Hamiltonian vector field again (cf Guillemin and Sternberg 1991). This is to say that there is a function $\{f, g\}$ on $F^a$ for which

$$[X_f, X_g] = X_{\{f,g\}}.$$

$\{f, g\}$ is called the Poisson bracket. $C^\infty(F^a, \mathbb{R})$ together with the bracket $\{,\}$ is a Lie algebra. To define the notion of a *quadratic polynomial* on $F^a$ we choose a unit vector $e_{\bar{q}} \in F^a$ and form $e_{\bar{q}} \cdot a =: e_{\bar{p}}$.

These two unit vectors define a coordinate system with coordinates $q$ and $p$. Obviously

$$\omega^a(e_{\bar{q}}, e_{\bar{p}}) = \langle e_{\bar{q}} \cdot a, e_{\bar{q}} \cdot a\rangle = \|e_q\|^2 = 1.$$

We now consider the collection $\mathcal{Q}$ of all homogenous quadratic polynomials in the internal variables expressed in coordinates $q$ and $p$ on $F^a$. Here each polynomial pol is identified with its real-valued function $f_{\mathrm{pol}}$ in two variables. The functions $f_{\frac{1}{2}q^2}$, $f_{\frac{1}{2}p^2}$ and $f_{pq}$ generate $\mathcal{Q}$. Obviously, $\mathcal{Q}$ is a sub-Poisson algebra of $C^\infty(F^a, \mathbb{R})$. For a polynomial $f \in \mathcal{Q}$ the principal part $a_f : F^a \to F^a$ of the vector field $X_f$ is a traceless linear isomorphism. Therefore, we have a map

$$\mathrm{ham} : \mathcal{Q} \longrightarrow \mathrm{sp}(F^a)$$

a Lie algebra isomorphism, given by

$$\mathrm{ham}(f_{\mathrm{pol}}) := X_{\mathrm{pol}}$$

where $f_{\mathrm{pol}}$ denotes a quadratic homogenous polynomial in $\mathcal{Q}$.

A *quantization* of the Poisson algebra $\mathcal{Q}$ of all quadratic homogenous polynomials in two variables is an irreducible representation on a Hilbert space. The quantization map $Q$ on $\mathcal{Q}$ is defined to be the composition

$$\mathcal{Q} \xrightarrow{\mathrm{ham}} \mathrm{sp}(F^a) \xrightarrow{\mathrm{i}\cdot\mathrm{d}U} \mathrm{End}(L^2(\mathbb{R}, \mathbb{C})).$$

The value of the representation $Q = \mathrm{i} \cdot \mathrm{d}U \circ \mathrm{ham}$ of any polynomial in $\mathcal{Q}$ is the quantization of that polynomial.

This quantization scheme can be extended to the semi-direct product of the metaplectic group with the Heisenberg group to yield a quantization for all inhomogeneous quadratic polynomials (cf Guillemin and Sternberg 1991).

In more detail, the quantization map $Q$ has a natural extension to the Poisson algebra $\mathcal{Q}_1$ of all *inhomogeneous* quadratic polynomials on $F^a$. In fact, ham can be extended to

$$\mathcal{Q}_1 \xrightarrow{\mathrm{ham}} \mathrm{sp}(F^a) \times \mathcal{G}^a$$

with the Heisenberg algebra $\mathcal{G}^a$ defined for some $a \in S^2$. The constant polynomial $1 \in \mathbb{R}$ and the linear ones $q$ and $p$ (the coordinate functions in $F^a$) are mapped onto the vectors $-a$, $-e_p$ and $e_q$, respectively. The Lie algebra $\mathrm{sp}(F^a) \times \mathcal{G}^a$ of the semi-direct product $\mathrm{Mp}(F^a) \times_s G^a$ is represented by $\mathrm{d}(\mathrm{i} \cdot U \circ \rho)$ where $\mathrm{i} \cdot U \circ \rho$ is defined by

$$\mathrm{i} \cdot U \circ \rho(A, g) := \mathrm{i} \cdot U(a) \circ \rho(g) \qquad \forall A \in \mathrm{Mp}(F^a) \quad \text{and} \quad \forall g \in G^a.$$

The composition of ham with the representation $\mathrm{d}(\mathrm{i} \cdot U \circ \rho)$ yields the desired extension of the quantization map (cf Guillemin and Sternberg 1991). Thus this quantization map is defined by means of $\rho$. The homogenous quadratic polynomials are quantized by means of the metaplectic representation which plays a fundamental role in geometric optics (cf Guillemin and Sternberg 1991, Gerrard and Burch 1994) on the one hand. On the other hand it relies on $\rho$ which plays a key role in time-frequency analysis (cf Folland 1989, Gröchenig 2000) as far as the ambiguity function or the (related) Wigner function is used. However, the Wigner function is also an important tool in the determination of light distributions in geometric optics (cf Brenner and Ojeda-Castaneda 1984, Bartelt *et al* 1980).

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